Chapter 5: Product measure.

There are two things to note:

Caratheodory: Show μ is a premeasure on semi rings

1) It's easier to work with semi rings a la Royden and apply Caratheodory extension for ∞ product measures
2) It's easiest to use MCL for finite product measure and Fubini-Tonelli. Royden does not use the Monotone Class lemma, and Le Gall does not even consider ∞ product measure.

↳ Showing something is true on a π-systems (closed under intersections)

$\sigma(\mathcal{E})$ ← π-system

Caratheodory.

Define: $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$. A meas rectangle $M = E \times F$, $E \in \mathcal{A}$ and $F \in \mathcal{B}$.

$\mu \times \nu(M) = \mu(E) \cdot \nu(F)$

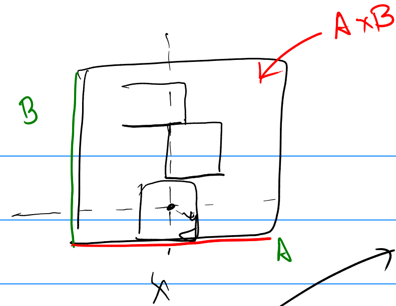
Prop: Let $\{A_k \times B_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable rectangles whose union also is a meas. rectangle $A \times B$. Then

$$\mu(A) \times \nu(B) = \sum_{k=1}^{\infty} \mu(A_k) \times \nu(B_k)$$

(μ is a pre-measure on meas. rectangles)

Pf: For any fixed $(x \in A)$

(x, y) belongs to exactly one $A_k \times B_k$ $\forall k$



follows from for fixed x, $x \in A$, $1_B(y) = \sum_{k=1}^{\infty} 1_{B_k}(y)$

$$1_A(x)\, 1_B(y) = \sum_{k=1}^{\infty} 1_{B_k}(y)\, 1_{A_k}(x)$$

$$\Rightarrow \quad 1_A(x)\, \nu(B) = \sum_{k=1}^{\infty} \nu(B_k)\, 1_{A_k}(x) \qquad \text{(by MON) for any fixed } x$$

Then integrating over μ, we have

$$\mu(A)\, \nu(B) = \sum_{k=1}^{\infty} \nu(B_k)\, \mu(A_k)$$

where in the last step we used MON to exchange limit and integral.

<u>Prop</u> Let $\mathcal{R}$ be the collection of meas. boxes in $X \times Y$ and for any $A \times B$, define

$$\mu \times \nu\, (A \times B) = \mu(A)\, \nu(B)$$

Then $\mu \times \nu$ is a premeasure & $\mathcal{R}$ is a semiring.
(finitely additive, countably monotone, $\mu \times \nu(\emptyset) = 0$)

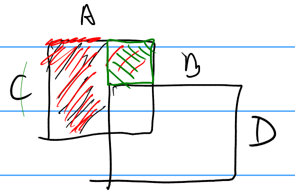
First show R is a semiring:

Pf: Let M_1, M_2 be meas. boxes. Then

$M_1 \cap M_2$ obviously a box. $M_1 \cap M_2 = A \cap B \times C \cap D$

Let $M_1 = A \times C$
$M_2 = B \times D$ Then $M_1 \setminus M_2$ also a disjoint

union of meas. rectangles.



Indeed $M_1 \setminus M_2 = A \setminus B \times C \sqcup A \cap B \times C \setminus D$

(Relative complements written as disjoint unions of meas. rect.)

Finite additivity of $\mu \times \nu$ & Previous lemma (If a box is a union of boxes)

$\mu \times \nu (A \times B) = \sum_{i=1}^{\infty} \mu(A_n) \nu(B_n)$

If E is covered by $\{E_n\}_{n=1}^{\infty}$ Left to establish monotonicity $E \subseteq \bigcup_{n=1}^{\infty} E_n = \bigsqcup_{n=1}^{\infty} E_k'$

Since R is a semiring $\{E_n\}$ can be written as a

disjoint union: E_1, $\underbrace{E_2 \setminus E_1}_{\text{disjoint union}}$, $E_3 \setminus (E_1 \cup E_2)$

$E_3 \cap (E_1^c \cap E_2^c) = \underbrace{(E_3 \setminus E_1)}_{\text{disjoint union}} \setminus E_2$ again disjoint union

So wlog $\{E_n\}$ is a disjoint union of meas. rectangles.

also measurable rectangles!

So $E = \bigsqcup_{k=1}^{\infty} (E \cap E_k)$ is a disjoint union of meas. rectangles

$$\mu\times\nu\,(E) = \sum_{k=1}^{\infty} \mu\times\nu\,(E\cap E_k) \le \sum_{k=1}^{\infty} \mu\times\nu\,(E_k)$$

$A\times B \subset C\times D$

$A\subset C \quad B\subset D$

monotonicity of $\mu\times\nu$ on meas. rectangles.

$\mu(A)\nu(B) \le \mu(C)\,\nu(D)$

But this is easy to check.

So we have a pre-meas on $\mathcal{R}$, the set of meas. rectangles.

So this produces a product meas. on the Caratheodory measurable sets, which is a complete σ-algebra. This method produces minor annoyances when it comes to Fubini and Tonelli.

This is because f that is $A\times B$ measurable is a larger collection of functions when compared to f that is $\sigma(A\times B)$ measurable.

Fubini: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be 2 meas. spaces and let

1) ν be complete

2) f be integrable on $X\times Y$ wrt $A\times B$

$A\times B \rightarrow$ σ-algebra of caratheodory meas. sets.

$\supset \sigma(R)$ $\hookrightarrow$ meas. rectangles.

Then for x a.e μ,

$f(x,\cdot)$ is integrable over Y wrt ν

and

$$\int f(x,y)\, d\mu\times\nu = \int \underbrace{\int f(x,y)\, d\nu}_{g(x) \to L^1(\mu)}\, d\mu$$

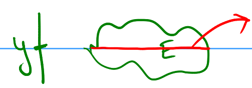
Other authors like Le Gall and Khoshnevisan define

the smallest σ-algebra containing the meas. rectangles is

to be the Product σ-algebra written $\mathcal{A} \times \mathcal{B} = \mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{R})$

Def: For any $E \subseteq X \times Y$ let

$E_y = \{ x : (x,y) \in E \}$ be the y-section of E



$E_y \subset X$

$\sigma(\mathcal{R})$

Lemma: If $E \in \mathcal{A} \times \mathcal{B}$, then $\forall y \in Y$, $E_y \in \mathcal{A}$.

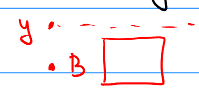
Pf: Fix $y \in Y$.

To show M_y is an MC

let $M_y = \{ E \in X \times Y : E_y \in \mathcal{A} \}$

1) M_y contains the π-system $\mathcal{R}$ of meas. rectangles.

$$(A \times B)_y = \begin{cases} A & \text{if } y \in B \\ \phi & \text{otherwise} \end{cases}$$

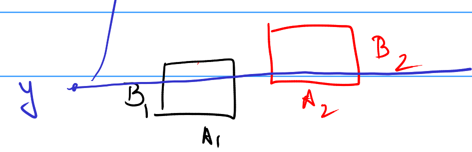


$\Rightarrow (A \times B)_y \in \mathcal{A}$

If $y \in B$ $A \in \mathcal{A}$, and if $y \notin B$ $(A \times B)_y = \phi$

2) It also contains finite disjoint unions of meas. rectangles.

$$\left(\bigsqcup_{i=1}^{n} A_i \times B_i \right)_y = \bigsqcup_{i : y \in B_i} A_i \in \mathcal{A}$$



3) Similarly if you have an increasing limit in M_y, then

$E_n \uparrow \in M_y$ st $(E_n)_y \in \mathcal{A}$

But $\left(\bigcup_n E_n\right)_y = \bigcup (E_n)_y \in \mathcal{A}$ since $\mathcal{A}$ is a σ-algebra.

If $x \in \left(\bigcup_n E_n\right)_y$ then $(x,y) \in \bigcup_n E_n \Rightarrow (x,y) \in E_n$ for some n

$\Rightarrow x \in (E_n)_y$.

~~Decreasing limits are similar and we're done:~~

Instead of decreasing limits, Le Gall requires: If $E \subset F \in M_y$ then $F \setminus E \in M_y$.

$(F)_y = \{x : (x,y) \in F\}$ $E_y = \{x : (x,y) \in E\}$

$$(F)_y \setminus (E)_y = \{x : (x,y) \in F, (x,y) \notin E\} = \{x : (x,y) \in F \setminus E\}$$
$$= (F \setminus E)_y$$

THEN

$M_y \supset \sigma(R)$ by MCL. This implies that if $E \in \underset{\mathcal{A} \otimes \mathcal{B}}{\overset{\|}{\sigma(R)}}$, then $E \in M_y$

$\Rightarrow E_y \in \mathcal{A}$ (true $\forall y$)

☆ LeGall just does it in one line: The set $M = \{C \in \mathcal{A} \times \mathcal{B} : C_y \in \mathcal{A}\}$ is a σ-algebra

Rem:

Le Gall also includes the extension to fns. Let $f : X \times Y \longrightarrow G$, where (G, Σ) is a meas. space be $\mathcal{A} \times \mathcal{B}$ meas.

Then $\forall y \in Y$, f_y is $\mathcal{B}$ meas: Let $D \in \Sigma$,

$f_y^{-1}(D) = \{x \in Y : f(x,y) \in D\} = (f^{-1}(D))_y \in \mathcal{B}$ by previous.

Lemma : Disintegration. Fix $E \in \mathcal{A} \times \mathcal{B} = \sigma(\mathcal{R})$. Let μ and ν be finite (or σ-finite) measures.

Since E_y is measurable, claim that

$y \mapsto \mu(E_y)$ is a $\mathcal{B}$ measurable function, — (★1)

$$\underbrace{(\mu \times \nu)(E) := \int \mu(E_y)\, d\nu(y)}_{\mu \times \nu(A \times B) = \mu(A)\nu(B),\ A\times B \in \mathcal{R}} \qquad \text{— (★2)}$$

is a (σ-) finite measure on $\mathcal{A} \times \mathcal{B}$

Pf:

(★2) true for all $E = A \times B \in \mathcal{R}$ since $E_y = A$ if $y \in B$.

$$E_y = \begin{cases} A & y \in B \\ \emptyset & \text{otherwise} \end{cases} \qquad \text{So}$$

$$\mu(E_y) = \mu(A) 1_B(y)$$

and $\displaystyle\int \mu(E_y)\, d\nu(y) = \int \mu(A) 1_B(y)\, d\nu(y) = \mu(A)\nu(B)$

$$\mathcal{M} = \{ E \in \mathcal{A} \otimes \mathcal{B} : \underbrace{★1}_{y \mapsto \mu(E_y) \text{ is } \mathcal{B} \text{ meas.}} \text{ and RHS of ★2 is well-defined} \}$$

$\mathcal{M}$ contains $\mathcal{R}$ by previous. $E \subset F \in \mathcal{M}$. Then $y \mapsto \mu(F_y)$ is $\mathcal{B}$ meas and so is $y \mapsto \mu(E_y)$. But $(F \setminus E)_y = F_y \setminus E_y$

$\mu((F \setminus E)_y) = \mu(F_y) - \mu(E_y)$ which is measurable, and is a difference of 2 $\mathcal{B}$ meas. functions

finiteness of μ for excision

$\int \mu((F \setminus E)_y)\, d\nu(y)$ is well-defined

If $E_n \uparrow \in \mathcal{M}$. $E = \bigcup E_n$, $E_y = \bigcup_n (E_n)_y \in \mathcal{A}$, $\mu(E_y) = \lim_{n \to \infty} \mu((E_n)_y)$ is defined everywhere on Y and is thus measurable.

($f_n \to f$ everywhere $\Rightarrow f$ is $\mathcal{B}$ meas.) (if $f_n \to f$ a.e., need $(Y, \mathcal{B})$ to be complete for f to be meas.)

If the limit didn't exist everywhere then $\mu(E_y)$ need not be Y measurable. (would need completeness for this)

So M is a monotone class and $\mu \times \nu(E) := \int \mu(E_y)\, d\nu(y)$

is well-defined and equal to $\mu(A) \times \nu(B)$ on R $(E = A \times B)$

To show $\mu \times \nu$ is a measure: Let $C = \bigsqcup_n C_n \in \mathcal{A} \otimes \mathcal{B}$.

$$\mu \times \nu(C) := \int \mu\left(\left(\bigsqcup C_n\right)_y\right) d\nu(y) = \int \sum \mu((C_n)_y)\, d\nu(y)$$

countable additivity of μ.

$$= \sum \int \mu((C_n)_y)\, d\nu(y) =: \sum \mu \times \nu(C_n)$$

linearity of ν integration

Hence its countably additive. Moreover, by Caratheodory its unique since its values are determined on R.

There is an easy extension to $(X_1 \times X_2 \cdots \times X_n, \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n, \mu_1 \otimes \cdots \otimes \mu_n)$ by induction.

What about $\bigotimes_{i=1}^{\infty} X_i$

<u>Fubini–Tonelli Theorem</u> If $f : X \times Y \to \mathbb{R}$ is product measurable, (f is $\sigma(R)$) then $\forall\ x \in X$, $y \mapsto f(x,y)$ is $\mathcal{B}$ measurable. If in addition

$f \in L^1(\mu \times \nu)$ then

$$x \longmapsto \int f(x,y)\,\nu(dy) \quad \text{and} \quad y \longmapsto \int f(x,y)\,\mu(dx)$$

are measurable and in $L^1(X, \mathcal{A}, \mu)$ and $L^1(Y, \mathcal{B}, \nu)$ respectively.

$$\int f \, d(\mu \times \nu) = \int \left[\int f(x,y)\,\nu(dy) \right] \mu(dx)$$

$$= \int \int f \, \mu(dx)\,\nu(dy)$$

☆ This can be skipped.

Pf: If $f = 1_E$ for $E \in \mathcal{A} \times \mathcal{B}$, we have what we need from the Lemma. By linearity this holds for simple fns. Finally we take limits (by truncating) and use Bounded and DOM to get what we need.

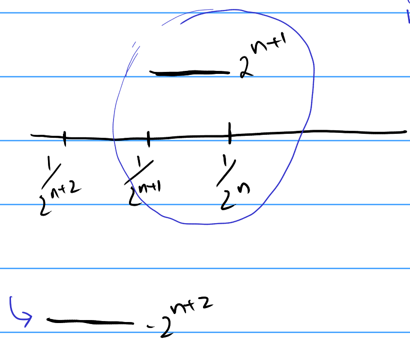
Cor: (Tonelli) If f is non-negative then we can replace BDD and DOM by MON, and REMOVE the $f \in L^1$ requirement.

Example that shows L^1 requirement.

Ex: Let $\psi_n = \begin{cases} 2^{n+1} & x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}] \\ 0 \end{cases}$

Let $f(x,y) = \sum_{n=0}^{\infty} \left[\psi_n(x) - \psi_{n+1}(x) \right] \psi_n(y) \quad \forall\, x,y \in (0,1]$

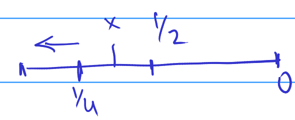
integrates to 0.



Picture of $\psi_n - \psi_{n+1}$

If $y \in (\frac{1}{2^{n+1}}, \frac{1}{2^n}]$ then $\int_0^1 f(x,y)\,dx = -2^{n+2}\frac{1}{2^{n+2}} + 2^{n+1}\frac{1}{2^{n+1}} = 0$

$$\int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} f(x,y)\,dy = \psi_n(x) - \psi_{n+1}(x)$$



Sum from 0 to $m-1$ to get $\int_{2^{m}}^{1} f(x,y)\,dy = \psi_0 - \psi_m(x)$ telescopes

But if $x > 0$ $\lim_{m\to\infty} \psi_m(x) = 0$

$$\Rightarrow \lim_{m\to\infty} \int_{2^{-m}}^{1} f(x,y)\,dy = \psi_0(x)$$

Integrate over all values of $x \in (0,1]$ to get

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0 \neq \int_0^1 \int_0^1 f(x,y)\,dy\,dx$$

$$= \int_0^1 \psi_0(x)\,dx = 1$$

The reason is that f is not absolutely integrable. (over $\mu \times \nu$)

Fubini-Tonelli also applies to σ-finite measures.

Another for Fubini to fail is if f is not product meas.

Dima Sinapova has an excellent article on the infinite ordinals and cardinals.
(Notices, vol 69, number 9 October 2022)

The ordinals are

$$0, 1, 2, 3, \ldots, \omega, \omega+1, \ldots$$

ω is the first ∞ ordinal #.

Well ordered set: S with a total ordering $<$ (or $\leq$) st every subset has a least element.

Ex: $\mathbb{N}$. $\mathbb{R}$ is not well ordered, ex $(1,2)$

Axiom of choice $\Leftrightarrow$ every set can be well-ordered (so you can just choose the least element)

Two well-ordered sets are order isomorphic if a bijection preserving their orderings.

Von Neumann definition: Ordinals provide canonical representatives of each well-ordered set.

Ex: Consider $(\{1, 2, \ldots\}, <)$. The ordinal is ω.

In Russell's Principia, ordinals are as equivalence classes of well-ordered sets (but this equivalence class is too large to be defined under ZF. Cantor knows what that means)

Von Neumann just provides a canonical representative for each ordinal.

$0 = \phi$ (empty set)

$1 = \{\phi\}$ $2 = \{1\} = \{\phi, \{\phi\}\}$ ...

$\omega = \{1, 2, \ldots\}$

"Each ordinal is the well-ordered set of all smaller ordinals"

Formally: A set S is an ordinal iff S is strictly well-ordered with respect to set membership, and every element of S is also a subset of S.

$2 = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}$ $\{\phi\} \subset \{\phi, \{\phi\}\}$

Is the class of all ordinals an ordinal? (Burali-Forti, similar to Russel's)

No it is not a set.

Each ordinal is also the set of ordinals before it. $\omega = \{1, 2, \cdots\} = \mathbb{N}$

In general $\beta = \{\alpha : \alpha < \beta\}$

So for each ordinal (viewed as a set) $(\beta, <)$ is a well order.

Transfinite recursion defines:

$\beta + 1 = \beta \cup \{\beta\}$ (successor stage)

Eg: ← succesor stage

$3 = 2 \cup \{2\}, \dots$

At the limit stage, one takes a union:

$\omega = \bigcup_{i \in \mathbb{N}} i$, $\omega + 1 =$

↑ axiom of union

$\underbrace{\omega, \omega+1, \dots}_{\text{transfinite recursion}}$, $\omega+\omega$ ← limit stage

$\omega + \omega = \omega \cup \{\omega + n \mid n < \omega\}$

How many axioms in ZF? 8

Need an alphabet (countably ∞) to represent sets.

$\neg, \wedge, \vee$

$\forall, \exists$

$=, \in, ( )$

1) Extensionality 2) Regularity
3) Schema of specification (restricts what sets you can construct)
4) Axiom of pairing 5) Axiom of union (union over elements of a set exist).
6) axiom schema of replacement 7) axiom of ∞. ∃ a set with ∞ly many members.
8) Axiom of power set $\boxed{9}$ Choice / well-ordering

There was this book I read a long time ago, about how the empty set was at the core of everything, and about Russel, and Frege and Logicomix.

Cardinal #s: We count using one-one correspondences.

So anyway, many ordinals are countable: $|\omega| = |\omega+1| \dots = |\omega \cdot \omega|$ and so on.

The first uncountable ordinal is ω_1 and by definition, this is not countable!
Then there is another jump at ω_2 and so on.

$\omega+1 = \{\omega\} \cup \{1, 2, \dots\}$ $\quad 1 = \{\phi\}$ $\quad 2 = \{1\} \cup \{\phi\} = \{\{\phi\}, \phi\}$

and so on.

Precisely, a cardinal is an ordinal κ which has no bijection to the ordinals below it.

The cardinality of a set A = $|\kappa|$ the cardinality of the ordinal (a set) equinumerous to it.

1-1 correspondence

Infinite cardinals: $\{\aleph_\alpha : \alpha \in \text{ordinal}\}$ $\quad \aleph_0$ is the cardinality of ω.

$\aleph_1$ is the cardinality of ω_1 and so on.

cartesian product

For $\kappa, \lambda \in \text{CARD}$, let $\kappa + \lambda = |\kappa \cup \lambda|$, $\kappa \cdot \lambda = |\kappa \times \lambda|$

Easy to see $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$

Def: let κ^λ be the cardinality of the set of functions from λ to κ.

Ex: $f : \kappa \to \{0,1\} = 2$ encodes a subset of κ in the usual way. We have

$2^\kappa = |P(\kappa)|$

Gen. Continuum hypothesis: For all ∞ cardinals κ, (let κ^+ be the successor cardinal of κ)

$2^\kappa = |P(\kappa)| = \kappa^+$

In particular $2^{\aleph_0} = \aleph_1$

By Cohen's method of forcing, the continuum (2^ω) can have arbitrary high cardinality! MAD!

(Gödel has a minimal model of ZFC where CH holds, Cohen proved $\exists$ models where 2^ω has arbitrary size $\Rightarrow$ CH must be indep. of ZFC)

Ex 5.11 (Importance of product measurability)

Let c be the 1st uncountable ordinal. Let S be all the ordinals smaller than c; S is Hartog's c-section of ordinals.

By continuum hypothesis, S has cardinality of continuum.

$\Rightarrow \exists \quad \phi : [0,1] \longrightarrow S \quad 1-1.$

ordering of ordinals

Let $E := \{ (x,y) \in [0,1]^2 : \phi(x) < \phi(y) \}$

For all $x \in [0,1]$ consider the x-section E_x of E.

$$E_x := \{ y \in [0,1] : (x,y) \in E \} = \{ y \in [0,1] : \phi(x) < \phi(y) \}$$

(all ordinals bigger than x)

E and E_x are nonempty since ϕ is a bijection.

all ordinals smaller than x

By definition E_x^c is countable (by definition of ordinals and ω_1)

all ordinals smaller than x. $\Rightarrow \mu(E_x) = 1$

Lebesgue

$$E_y = \{ x \in [0,1] : (x,y) \in E \} = \{ x \in [0,1] : \phi(x) < \phi(y) \}$$

Obviously E_y is countable, $\mu(E_y) = 0$

$$\int_0^1 \mu(E_x)\,\mu(dx) = 1 \neq 0 = \int_0^1 \mu(E_y)\,\mu(dy)$$

1   0

bounded function on a finite meas. space.

$\Rightarrow \int 1_E \, dx\,dy \neq$ either of the above, so it must mean that E is not product measurable.

This is another example of a nonmeasurable set!

Infinite product measures.

We now know how to construct product measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. In fact, we can do this on any topological space.

What is this?

What about the space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$?

Defn: Given a topological space Ω, we define a cylinder set in Ω^∞ to be

$A : A_1 \times A_2 \times A_3 \ldots$, where A_i are open sets.

such that for some N, $A_n = \Omega$ for $n \geq N$.

The smallest such N is called the dimension of A: $\dim(A) = N$.

Remark: We should not work with General Topological spaces here like Khoshnevisan. This is because to prove things with some generality we need a more general version of Caratheodory-Hahn for topological spaces!

For completeness I will state it w/o defining the terms.

Let (X, T) be a Topological space. μ is a Borel measure on $\mathcal{B}(T)$ if

1) $\mu(K) < \infty \quad \forall K$ compact.

A Borel measure is called Radon if

2) $\mu(E) = \inf \{ \mu(U) : U$ is a neighborhood (open set) of $E \}$

3) $\mu(E) = \sup \{ \mu(K) : K$ is a compact set $K \subset E \}$

Lebesgue measure on $\mathbb{R}^n$ is a Radon measure. (saw this in your HW)

Let $(X, \mathcal{T})$ be a topological space. A pre-measure $\mu : \mathcal{T} \to [0, \infty]$ is called Radon if

1) $\mu(U) < \infty$ if $\bar{U}$ is compact $(U \in \mathcal{T})$

2) $\mu(O) = \sup\{\mu(U) : \bar{U} \subseteq O, U \in \mathcal{T}\}$

↑ using only open sets here

In some sense, it's saying the boundary of open sets has very little meas.

Theorem: Let $(X, \mathcal{T})$ be a locally compact Hausdorff space, and μ a Radon pre-measure. Then the restriction of μ^*, the induced outer measure, to $\mathcal{B}(X)$ (the smallest σ-algebra that contains $\mathcal{T}$) is a measure that extends μ. (Caratheodory-Hahn)

Remark: Sevah proves Kolmogorov's theorem assuming the P^n is a family of Radon measures on a topological space. Again he proves everything "by hand" rather than appealing to Caratheodory-Hahn like we have done.

Remark: We will not prove the general theorem here, but simply prove it for $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ defined below. This will allow us to check the "semi-ring" property easily.

☆ Ex: Prove that a topology $\mathcal{T}$ on a Hausdorff space is a semiring iff it is discrete.

The general theory can be found in Royden Ch 21.

"Our" cylinder sets: (Durrett 5a, Theorem A.3.1)

We say A is a cylinder set if $\Omega = \mathbb{R}$.

$$A = \prod_{i=1}^{n} A_i \times \prod_{i=n+1}^{\infty} \mathbb{R} \qquad \text{where } A_i = (a_i, b_i]$$

It's clear that $\{\pi_n(A) : A \text{ is a cylinder set}\}$ generates $\mathcal{B}(\mathbb{R}^n)$

Defn: Let $\mathcal{B}^\infty$ be the smallest σ-algebra containing all the cylinder sets. We will henceforth use the shorthand $\mathcal{B}^n$ for $\mathcal{B}(\mathbb{R}^n)$.

Ex: $I_\ell = [0, \frac{1}{\ell})$ $\quad \ell = 1, 2, 3$

$$\text{Leb}^\infty\left(\prod_{\ell=1}^{3} I_\ell \times \prod_{\ell=4}^{\infty} [0,1)\right) = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3}$$

Define $\pi_n : \Omega^m \longrightarrow \Omega^n$ as $x \longmapsto (x_1, \ldots, x_n)$
for any $m \geq n$. m can also be ∞. (PROJECTION)

A family of probability spaces $\{(\Omega^n, \mathcal{B}^n, P^n)\}$ is called consistent if
$$P^m(\pi_n(A) \times \Omega_{n+1} \cdots \times \Omega_m) = P^n(\pi_n(A)) \qquad \infty > m \geq n$$
for any cylinder set A.

☆ Ex: Show that this is equivalent to the condition

$$P^{n+1}(A_1 \times A_2 \times \cdots \times A_n \times \Omega_{n+1}) = P^n(A_1 \times A_2 \cdots \times A_n)$$

Rem: P^n are not necessarily product measures.

Let $(\mathbb{R}, \mathcal{B}, \nu)$ be counting measure. Is it outer regular?

$\nu(\{1\}) = 1$ But every $\nu((a,b)) = +\infty$. Not outer regular.

But it is inner regular (Ex).

There are a few more wikipedia examples. Let $X = \{x : x \leq \Omega\}$ where Ω is the 1st uncountable ordinal.

$T = \{(a,b) : a, b \in X\}$ "open intervals".

Show (X, T) is a compact Hausdorff space. If $A \in \mathcal{B}(T)$ and $A \supset$ an unbounded closed subset of the ordinals, $\mu(A) = 1$ For example if $A \supset \{\phi, 1, 2, \cdots, \omega\}$. Otherwise $\mu(A) = 0$. This is neither inner nor outer regular.

<u>Theorem</u>: (Kolmogorov). Suppose P^n is a consistent family of Radon measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

Then $\exists !$ measure P^∞ on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ such that for all cylinder sets A,

$$P^\infty(A) = P^n(\pi_n(A)) \quad \forall n, \text{ and } A \text{ st } \dim(A) = n$$

Pf: It follows the same strategy as before:

R = {set of all finite dim cylinder sets}

Is R a semiring? Let $A, B \in R$ and $n = \max(\dim(A), \dim(B))$

Then $\pi_n(A \setminus B) = \bigsqcup_{i=1}^{k} C_i$ where $\dim(C_i) = n$.

Thus $A \setminus B = \bigsqcup_{i=1}^{k} \pi_n^{-1}(C_i)$

Let $P^\infty(A) = P^n(A)$ for any $A \in R$ $\dim(A) = n$.

This is well-defined, by consistency.

Its enough to show P^{∞} is finitely additive on R.

Let $\bigsqcup_{i=1}^{\infty} A_i = A \in R$ Then let $M = \dim(A)$.

To show $P^{\infty}(A) = \sum_{i=1}^{\infty} P^{\infty}(A_i)$

If $\sup_i \dim(A_i) < M$, this is easy! The issue is that $\dim(A_i) \to \infty$ if $i \to \infty$. In particular FINITE additivity is just inherited.

Note that $\dim(A_i) < \infty$. Fix N and let $n = \max_{i=1,\dots,N}(\dim(A_i))$

Then $\pi^n(A) \supseteq \bigsqcup_{i=1}^{N} \pi^n(A_i)$ This is because if $x \in A_i$ then $x \in A$. Every $y \in \pi^n(A_i)$ is of the form $\pi^n(x)$ for some $x \in A_i$
$\Rightarrow \pi^n(x) \in A$.

Then since P^n is a meas.

$$P^n(\pi^n(A)) \geq \sum_{i=1}^{N} P^n(\pi^n(A_i)) = \sum_{i=1}^{N} P^{\infty}(A_i)$$

Let $n \to \infty$ to get $P^{\infty}(A) \geq \sum_{i=1}^{N} P^{\infty}(A_i)$ Now let $N \to \infty$

So this implies $\sum_{i=1}^{\infty} P^{\infty}(A_i)$ is summable.

let $\dim(A) = k$

Now consider $K \subset \Pi^k(A)$ where K is compact and $P^k(K) \geq P^k(A) - \epsilon$ (Inner regularity here), and $\dim(K) = k$.

Let $O_i \supset \pi^{n(i)}(A_i)$, $\dim(O_i) = \dim(A_i)$ be open st (outer regularity of P^n)

$$P^{n(i)}(O_i) \leq P^{n(i)}(\pi^{n(i)}(A_i)) + \frac{\epsilon}{2^i}$$

here $n(i) = \dim(A_i)$

Now, let $\hat{K} = K \times [-M, M] \times [-M, M] \cdots\cdots$

$\tilde{K}$ is a product of compact sets and hence its compact in the product topology.

$\Pi^{-1}_{n(i)}(O_i)$ are sets in $\mathbb{R}^\infty$ that are open. (The coordinate fns are continuous in the product topology, so it must pull back open sets to open sets)

Thus $\tilde{K} \subset \bigcup_{i=1}^{\infty} \pi^{-1}_{n(i)}(O_i)$ If $x \in \tilde{K}$ $\pi^k(x) \in K \Rightarrow x \in A$

$\Rightarrow$ $x \in \Pi^{-1}_{n(i)}(O_i)$. Therefore $\exists$ a finite subcover of $\tilde{K}$,

$$\tilde{K} \subset \bigcup_{i=1}^{N} \Pi^{-1}_{n(i)}(O_i) \quad \text{(after relabeling the } O_i)$$

Now these sets $\Pi^{-1}_{n(i)}(O_i)$ are cylinder sets, let $n = \max(n(1) \ldots n(N), k)$

$$\Pi_n(\tilde{K}) \subset \bigcup_{i=1}^{N} \pi_n(\pi^{-1}_{n(i)}(O_i))$$

and $$P^n(\pi_n(\tilde{K})) \leq \left(\sum_{i=1}^{N} P^n(O_i)\right) \leq \sum_{i=1}^{N} P^\infty(A_i) + \epsilon$$

$\dim(K) = k \leq n$ and $\pi_n(\tilde{K}) = K \times \underbrace{[-M,M] \times \cdots [-M,M]}_{n-k}$

$\subseteq K \times \mathbb{R} \times \cdots \times \mathbb{R}$ But M can be chosen so large st

$P^n(\pi_n(\hat{K})) \geq P^k(K) - \epsilon \geq P^k(A) - 2\epsilon = P^\infty(A) - 2\epsilon$.

Putting this together gives $P^\infty(A) \leq \sum_{i=1}^{N} P^\infty(A_i) + 3\epsilon$

Let $N \to \infty$, and since ϵ is arbitrary this gives us what we need.

<u>Remark:</u> This is identical to the route we took with Lebesgue measure.

Remark: Nevah's proof only seems to care about inner regularity.

Remark: We proved (or gave as HW) that (E, d) is a metric space & μ is a finite meas on $\mathcal{B}(E)$, then its both inner and outer regular.

So this shows Durrett's version of consistency: Let (S_i, d_i) be a sequence of Polish spaces, and $\Omega = \otimes S_i$

Suppose μ is finite, inner regular and (X,T) is topological. Let μ^* be the outer meas induced by μ restricted to T. Recall the definition of Radon-premeasure.

Let (X,T) be a topological space. A pre-measure $\mu : T \to [0,\infty]$ is called Radon if

1) $\mu(U) < \infty$ if $\bar{U}$ is compact $(U \in T)$

2) $\mu(O) = \sup\{\mu(U) : \bar{U} \subseteq O,\ U \in T\}$

μ being finite implies 1). Inner regularity implies 2).

<u>Theorem</u>: Let (X,T) be a locally compact Hausdorff space, and μ a Radon pre-measure. Then the restriction of μ^*, the induced outer measure, to $\mathcal{B}(X)$ (the smallest σ-algebra that contains T) is a measure that extends μ. (Caratheodory-Hahn)

By Caratheodory-Hahn, $\mu^*\big|_{\mathcal{B}(T)} = \mu$ and thus $\mu^*(A) = \mu(A) = \inf\{\mu(U) : U \in T\}$ is outer regular.

☆Exercise.